On Moessner’s theorem

Dexter Kozen\textsuperscript{1} \quad Alexandra Silva\textsuperscript{2,3}

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\textsuperscript{2}Radboud Universiteit Nijmegen, The Netherlands
\textsuperscript{3}HasLab, Universidade do Minho, Portugal

QAIS workshop, October 2011
Moessner’s construction (n=4)

|   | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  | 9  | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 |
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| 2 | 3  | 6  | 11 | 17 | 24 | 33 | 43 | 54 | 67 | 81 | 96 | 113| 131| 150| 171| 193| 216| 242| 271| 303| 337| 373| 411|
| 3 | 4  | 15 | 32 | 65 | 108| 175| 256| 369| 500| 625| 774| 946| 1131|1336|1561|1806|2071|2366|2690|3045|3423|3833|
| 4 | 16 | 81 | 256| 625|1296|256|512|1024|2048|4096|8192|16384|32768|65536|131072|262144|524288|1048576|2097152|4194304|8388608|16777216|

Moessner’s Conjecture/Theorem

Works for all $n \in \mathbb{N}$
Moessner’s construction (n=4)

Moessner’s Conjecture/Theorem
Works for all $n \in \mathbb{N}$
Moessner’s construction (n=4)

|   | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  | 9  | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 |
|---|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|
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| 2 | 1  | 3  | 6  | 11 | 17 | 24 | 33 | 43 | 54 | 67 | 81 | 96 | 113| 131| 150| 171| 193| 216|
| 3 | 1  | 4  | 15 | 32 | 65 | 108| 175| 256| 369| 500| 671| 864|
| 4 | 1  | 16 | 81 | 256| 625| 1296|

Moessner’s Conjecture/Theorem

Works for all $n \in \mathbb{N}$
Moessner’s construction (n=4)

\[ \begin{array}{ccccccccccccccccccc}
1 & 16 & 81 & 256 & 625 & 1296 & 1681 & 2041 & 2449 & 2904 & 3396 & 3939 & 4521 & 5145 & 5805 & 6495 & 7215 & 7961 & 8735 & 9537 \\
1 & 2^4 & 3^4 & 4^4 & 5^4 & 6^4 \\
\end{array} \]

Moessner’s Conjecture/Theorem

Works for all \( n \in \mathbb{N} \)

Alexandra Silva (RUN)
Moessner’s construction (n=4)

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Moessner’s Conjecture/Theorem
Works for all \( n \in \mathbb{N} \)

Alexandra Silva  (RUN)  
On Moessner’s theorem  
QAIS  2 / 18
Moessner’s construction (n=4)

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$1^4, 2^4, 3^4, 4^4, 5^4, 6^4$

Moessner’s Conjecture/Theorem

Works for all $n \in \mathbb{N}$
History

1951 Moessner conjectures it

1952 Perron proves it

1952 Paasche and Salie generalize it

1966 Long presents and alternative proof (and generalizes it)

2010 Hinze, Rutten&Niqui present new proofs of Moessner’s theorem

2011 This talk: an uniform proof of all the theorems
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Paasche asked: what if we cross out 1, 3, 6, 10, ...?

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We get the factorials: 1, 2, 6, 24, 120, ... = 1!, 2!, 3!, 4!, 5!, ....
Paasche asked: what if we cross out 1, 3, 6, 10, ...?

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</tr>
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We get the factorials: 1, 2, 6, 24, 120, . . . = 1!, 2!, 3!, 4!, 5!, . . . .
And the magic continues... 

What if we increment the increment by one in each step?

<table>
<thead>
<tr>
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What is the sequence

1, 2, 12, 288, ... ?

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1, 2, 12, 288, ... = 1!, 2!1!, 3!2!1!, 4!3!2!1!, ... = 1!!, 2!!, 3!!, 4!!, ...
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1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 ...
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2 13 12 28 51 82 103 133 196 272 352 ...
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An alternative procedure

Long's observation
First triangle: Pascal triangle; all have Pascal property

Long's procedure
starting point: Pascal triangle
next step: consider the $n$th northeast-to-southwest row. Take prefix sums and make that the first column, and let the first row be a sequence of 1’s. Complete the triangle using the Pascal property.
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Yet another generalization (Long & Salie)

What if instead of the natural numbers we start with

\[ a, a+d, a+2d, a+3d, a+4d, a+5d, a+6d, a+7d, a+8d, a+9d, a+10d, a+11d, \ldots \]

\[ a, a+d, a+2d, a+3d, a+4d, a+5d, a+6d, a+7d, a+8d, a+9d, a+10d, a+11d, \ldots \]

Given \( n \in \mathbb{N} \), the Moessner construction yields

\[ a \cdot 1^{n-1}, (a + d) \cdot 2^{n-1}, (a + 2d) \cdot 3^{n-1}, \ldots \]

when starting from the sequence \( a, a + d, a + 2d, \ldots \)
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\[ a \quad 2a+d \quad 3a+3d \quad 4a+7d \quad 5a+12d \quad 6a+18d \quad 7a+26d \quad 8a+35d \quad 9a+45d \]

\[ a \quad 3a+d \quad 7a+8d \quad 12a+20d \quad 19a+46d \quad 27a+81d \]

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Proofs

- The proofs of Perron, Paasche, Long and Salie all have in common the manipulation of binomial coefficients.
- Hinze’s proof (2010) involves calculations scans (FP).
- Rutten’s proof is coinductive.
- Not obvious if the last two can be generalized.
Our view on Moessner’s theorem

- we take Long’s triangle view
- we describe the process as operations on formal power series on two variables
- this yields a proof of Moessner’s theorem and its generalizations all at once!
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The Pascal triangle $\Delta = \Delta(x, y)$ is

$$\Delta(x, y) = \frac{1}{1 - (x + y)} = \sum_{m=0}^{\infty} (x + y)^m = \sum_{i,j} \binom{i + j}{i} x^i y^j. \quad (1)$$
Moessner’s construction, algebraically

The “$n$th northeast-to-southwest row” of $p \in \mathbb{Z}(x, y)$ is the homogeneous component of degree $n$, denoted $[p]_n$.

The operation of “taking prefix sums” is multiplying by $\frac{1}{1-x} = \sum_{i=0}^{\infty} x^i$. 
Sequences of triangles

Each successive level-$n$ Moessner triangle is obtained from the previous by taking the homogeneous component of degree $n$, evaluating at $y = 1$, and multiplying by $\Delta$.

We define inductively

$$h_0(x, y) = 1 \quad h_{k+1}(x, y) = [h_k(x, 1) \cdot \Delta(x, y)]_n,$$

then the $k$th level-$n$ Moessner triangle is $h_k(x, 1) \cdot \Delta$ and the final sequence in the Moessner construction is the lead coefficient of $h_k(x, 1)$ for $k = 1, 2, 3, \ldots$. 
Instead of $h_0(x, y) = 1$, we can take $h_0 \in \mathbb{Z}[x, y]$ arbitrary (Salie’s generalization).

For Paasche’s construction we need to take homogeneous components not of a fixed $n$ but of an arbitrary increasing sequence.

Let $d(0), d(1), d(2), \ldots$ of nonnegative integers and $n(k) = \sum_{i=0}^{k} d(i)$. The $n(k)$’s are the positions one should delete.

For Moessner

\[
\begin{array}{cccccc}
  d(0) & d(1) & d(2) & d(3) & \cdots \\
  n & 0 & 0 & 0 & \cdots \\
  n & n & n & n & \cdots \\
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\]
Define inductively

\[ h_{k+1}(x, y) = [h_k(x, 1) \cdot \Delta(x, y)]_{n(k+1)}. \] (2)

The Moessner construction is the special case \( h_0 = 1, \) \( d(0) = n, \) and \( d(i) = 0 \) for \( i \geq 1. \)
Main theorem

Theorem

Let \( h_k \) be the sequence defined by (2). For all \( k \geq 0 \),

\[
h_k(x, y) = \prod_{i=0}^{k-1} ((k - i)x + y)^{d(i)} \cdot h_0(x, kx + y).
\]
Paasche’s, Long’s, and Moessner’s theorems are now immediate consequences of Theorem 1.

**Corollary (Moessner’s Theorem)**

If \( h_0 = 1 \), \( d(0) = n \), and \( d(k) = 0 \) for \( k \geq 1 \), then the lead coefficient of \( h_k(x, 1) \) is \( k^n \) for all \( k \geq 1 \).
Paasche’s, Long’s, and Moessner’s theorems are now immediate consequences of Theorem 1.

**Corollary (Long’s Theorem)**

If $h_0 = (a - d)x + dy$, $d(0) = n - 1$, and $d(k) = 0$ for $k \geq 1$, then the lead coefficient of $h_k(x, 1)$ is $(a + (k - 1)d)k^{n-1}$ for all $k \geq 1$. 
Paasche’s, Long’s, and Moessner’s theorems are now immediate consequences of Theorem 1.

**Corollary (Paasche’s Theorem)**

For $h_0 = 1$ and any sequence $d$, the lead coefficient of $h_k(x, 1)$ is

$$
\prod_{i=0}^{k-1} (k - i)^{d(i)}
$$

for all $k \geq 0$. In particular, the sequences $d = 1, 1, 1, \ldots$ and $d = 1, 2, 3, \ldots$ yield the factorials and superfactorials, respectively.
Conclusions

- First proof that covers all the generalizations
- Proofs have a striking simplicity (no binomial coefficient manipulations!)
- Opens the door to new Moessner-like theorems (multidimensional generalization).
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Thank you for your attention!