Generalized Determinizations via Coalgebras

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Motivation (by example)

- Coalgebras are a suitable framework to study the behaviour of dynamical systems.
- Much of the coalgebraic approach can be nicely illustrated with deterministic automata.

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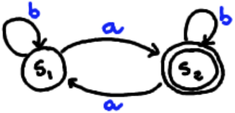

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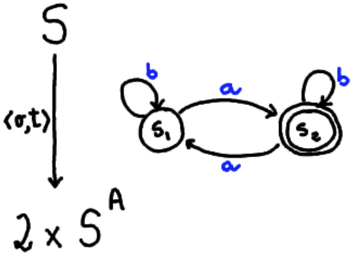

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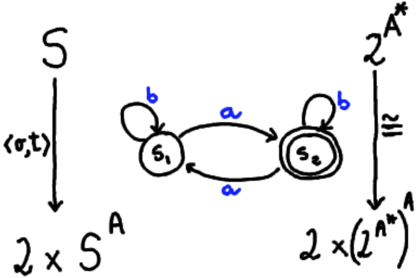

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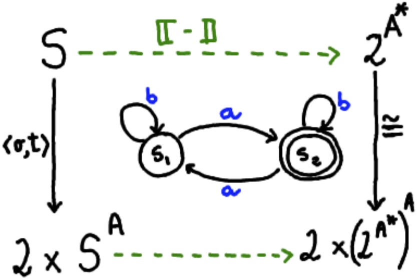

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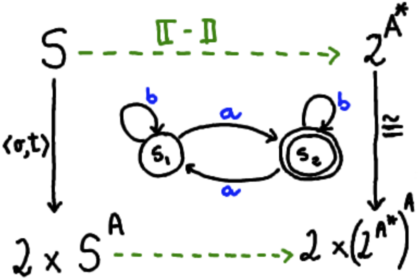
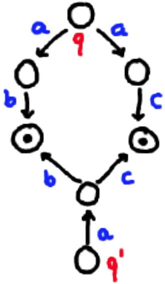


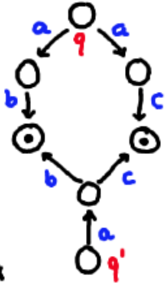
$$[\![s_1]\!] = \{ w \mid |w|_a \text{ is odd} \}$$

Motivation (by example, cont.)



Motivation (by example, cont.)

$$S \xrightarrow{\langle o,t\rangle} 2 \times \mathcal{P}_\omega(S)^A$$



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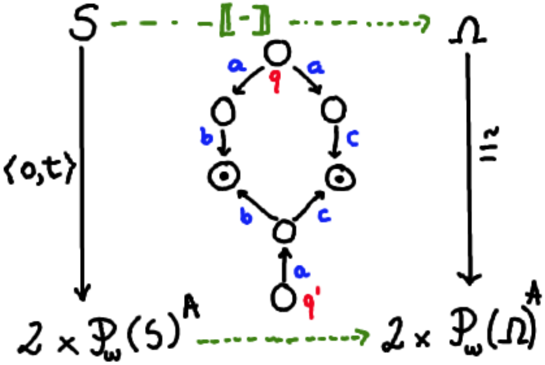

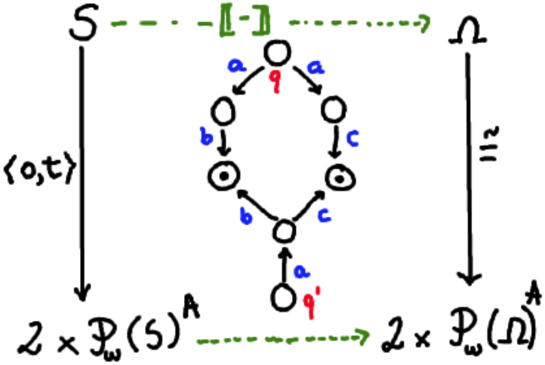

$[\![q]\!] \neq [\![q']\!]$ (different branching structure)

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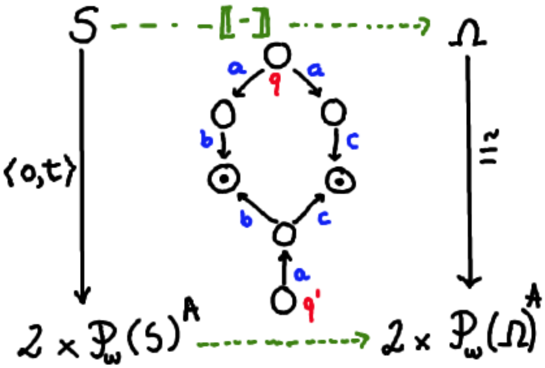

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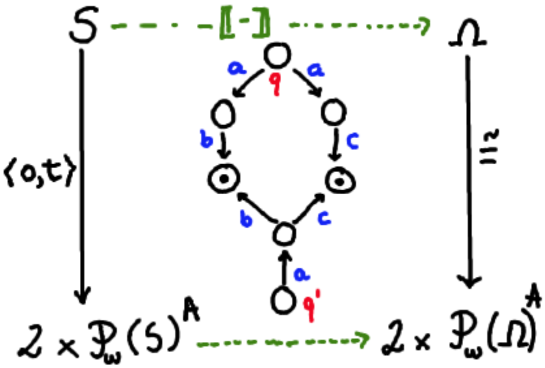

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How do we study NDA wrt language equivalence?

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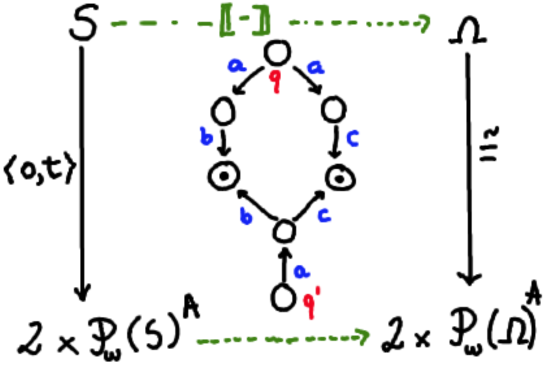

$[\![q]\!] \neq [\![q']\!]$ (different branching structure) but: $L_q = L_{q'} = \{ab, ac\}$

How do we study NDA wrt language equivalence?

Turn a non deterministic automaton into a deterministic one via the *powerset construction* and then apply usual semantics.

Example I: Determinizing (coalgebraically)

$$\begin{tikzcd} S \arrow[d, "\langle o,t \rangle"] \\ 2 \times \mathcal{P}_\omega(S)^A \end{tikzcd}$$

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$$\overline{o}(Q) = \begin{cases} 1 & \exists_{q \in Q} o(q) = 1 \\ 0 & \text{otherwise} \end{cases} \qquad \overline{t}(Q)(a) = \bigcup_{q \in Q} t(q)(a)$$

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$$L_s = \llbracket \{s\} \rrbracket$$

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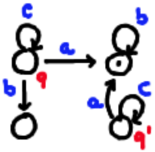

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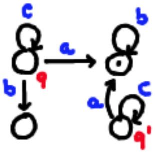

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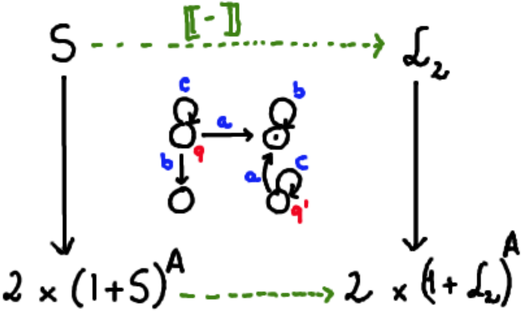

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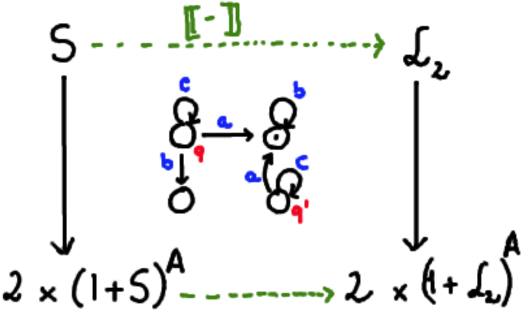

$\mathcal{L}_2$ are pairs of languages $\langle V, W \rangle$ (<accepted words, domain>)

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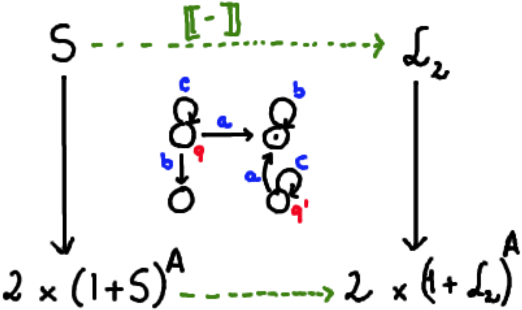

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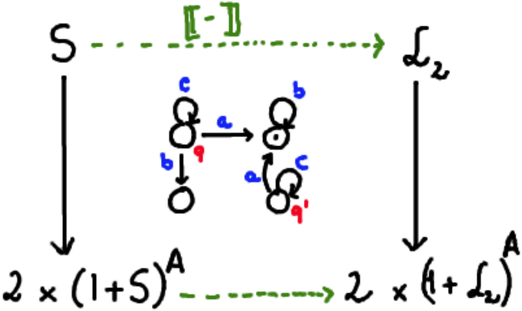

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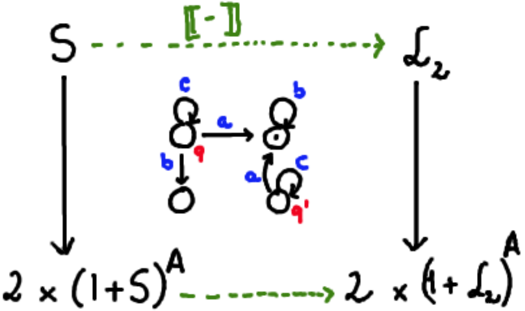

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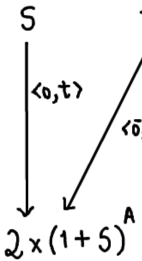

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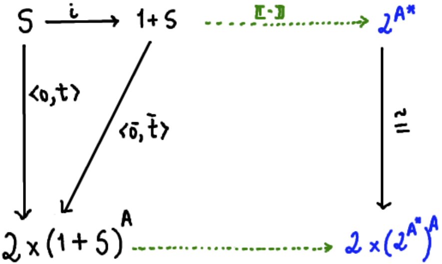

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S \arrow[r, "i"] \arrow[d, "\langle o,t \rangle"'] & 1+S \arrow[r, dashed, "[\![-]\!]"] \arrow[dl, "\langle \bar{o},\bar{t} \rangle"] & 2^{A^*} \arrow[d, "\cong"] \\
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Transform an FT-coalgebra (X,f) into an F-coalgebra ($T(X)$, $f^\sharp$).

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Transform an FT-coalgebra (X,f) into an F-coalgebra $(T(X), f^\sharp)$.
If F has final coalgebra: $x_1 \approx_F^T x_2 \Leftrightarrow [\![\, \eta_X(x_1) \,]\!] = [\![\, \eta_X(x_2) \,]\!]$.

In a nutshell. . .

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Ingredients:

- A monad T;
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Ingredients:

- A monad T;
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- An extension $f^\sharp$ of f; We can require $FT(X)$ to be a T-algebra: $(FT(X), h\colon T(FT(X)) \to FT(X))$

$$f^\sharp\colon\ T(X) \xrightarrow{T(f)} T(F(T(X))) \xrightarrow{h} F(T(X))$$

Examples revisited

NFA $F(X) = 2 \times X^A$, $T = \mathcal{P}$, $2 \times \mathcal{P}(X)^A$ is a join-semilattice;

PA $F(X) = 2 \times X^A$, $T = 1 + -$, $2 \times (1 + X)^A$ is a pointed set.

What is the relation between $\approx_F^T$ and $\sim_F$?

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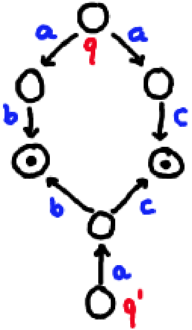

$q \approx^T_F q'$

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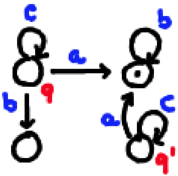

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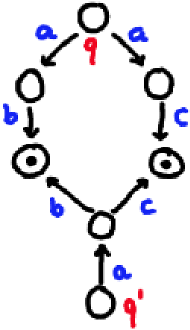

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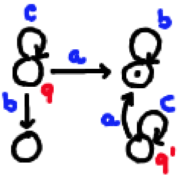

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Bisimilarity implies linear bisimilarity

Theorem

$$\sim_F \;\Rightarrow\; \approx_F^T$$

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Theorem

$$\sim_F \;\Rightarrow\; \approx_F^T$$

The above theorem instantiates to well known facts:

- for NDA ($F(X) = 2 \times X^A$, $T = \mathcal{P}$) that bisimilarity implies language equivalence;
- for PA ($F(X) = 2 \times X^A$, $T = 1 + -$) that equivalences of pair of languages, consisting of defined paths and accepted words, implies equivalence of accepted words;
- for probabilistic automata ($F(X) = [0,1] \times X^A$, $T = \mathcal{D}_\omega$) that probabilistic bisimilarity implies weighted language equivalence.

Examples, Examples, Examples,...

- **Partial Mealy machines** $S \to (B \times (1+S))^A$;
- **Automata with exceptions** $S \to 2 \times (E+S)^A$;
- **Automata with side effects** $S \to E^E \times ((E \times S)^E)^A$;
- **Total subsequential transducers** $S \to O^* \times (O^* \times S)^A$;
- **Probabilistic automata** $S \to [0,1] \times (\mathcal{D}_\omega(X))^A$;
- **Weighted automata** $S \to \mathbb{R} \times (\mathbb{R}_\omega^X)^A$;
- ...

Conclusions

- Lifted *powerset construction* to the more general framework of *FT*-coalgebras;
- Uniform treatment of several types of automata, recovery of known constructions/results;
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