Pre-Galois Connection on Coalgebras for Generic Component Refinement

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Component based software development as a promising paradigm to deal with the increasing complexity in software design.

Components must be specified and implemented before it can be analyzed and used.

Coalgebras can be used as a mathematical model for components.

Galois connection has been widely used to ensure the correctness of refinement relations.

Do we have a notion like Galois connections between coalgebras to witness refinement of components?
Motivation

- Component based software development as a promising paradigm to deal with the increasing complexity in software design.
- Components must be specified and implemented before it can be analyzed and used.
- **Coalgebras** can be used as a mathematical model for components.
- **Galois connection** has been widely used to ensure the correctness of refinement relations.
- Do we have a notion like Galois connections between coalgebras to witness refinement of components?
What will we show?

We will show how to...

- ... unify the behavior model and transition types into one functor over the Kleisli category for the coalgebra model of components
- ... rebuild refinement relationship between coalgebraic structures
- ... use pre-Galois connection in reasoning about refinement of components
Generic Components

Components can be specified in a generic way which means that the underlying behavior model is taken as a specification parameter, and abstracted to a monad $B$.

Some useful possibilities:

- **Identity**, $B = \text{Id}$, which retrieves the simple total and deterministic behavior.

- **Partiality**, $B = \text{Id} + 1$, i.e., the maybe monad, capturing the partial behavior which describes the possibility of deadlock or failure.

- **Non-determinism**, $B = \mathcal{P}$, modeling the non-deterministic branching behavior.
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Generic Components

The type of state transitions of the component is described by a functor $T$. For example, if we take $I$ and $O$ be sets acting as component input and output interfaces, then $T$ can be defined as the $\textbf{Set}$ endofunctor

$$T = (\text{Id} \times O)'$$

A state-based component can be modeled as a pointed coalgebra $(u \in U, \alpha : U \rightarrow BTU)$ in $\textbf{Set}$ with

- $B$ a monad,
- $T$ a functor,
- a distributive law $TB \Rightarrow BT$ implicit, that describes the way how $B$’s effect is distributed over the transition type represented by $T$,
- the point $u$ being taken as the “initial” or “seed” state.
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- the point $u$ being taken as the “initial” or “seed” state.
For each monad \( B \) on \( \textbf{Set} \), the Kleisli category for \( B \), denoted by \( \mathcal{K}(B) \), can be constructed as follows:

- Objects in \( \mathcal{K}(B) \) are the same as in \( \textbf{Set} \). They are just sets.
- An arrow \( U \to V \) in \( \mathcal{K}(B) \) is a function \( U \to BV \) in \( \textbf{Set} \).
- Composition of arrows in \( \mathcal{K}(B) \) is defined using multiplication \( \mu_U : BBU \to BU \).
- Identity arrow \( \text{id} : U \to U \) in \( \mathcal{K}(B) \) is the unit \( \eta_U : U \to BU \) in \( \textbf{Set} \).

The functor \( T \) can be lifted to a functor \( \mathcal{K}(T) \) on the Kleisli category \( \mathcal{K}(B) \) via the distributive law.

Considering the component model, a component is just a pointed coalgebra \( (u \in U, \alpha : U \to \mathcal{K}(T)U) \) in the Kleisli category \( \mathcal{K}(B) \).
Kleisli Category

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Order

- For a Kleisli category $\mathcal{K}(B)$ and any functor $T$, an order $\leq_{\mathcal{K}(T)}$ on $\mathcal{K}(T)$ is defined as a collection of preorders $\leq_{BTU} \subseteq BTU \times BTU$, for each set $U$, such that the following diagram commutes:

\[
\begin{array}{c}
\text{PreOrd} \\
\downarrow \\
\mathcal{K}(B) \xrightarrow{\leq_{\mathcal{K}(T)}} \mathcal{K}(B)
\end{array}
\quad \downarrow
\quad \begin{array}{c}
(\mathcal{K}(T), \leq_{\mathcal{K}(T)}) \\
\downarrow
\end{array}
\quad \begin{array}{c}
U \quad \quad \rightarrow
\end{array}
\quad \begin{array}{c}
BTU
\end{array}
\]

- and for any $f : U \rightarrow V$, $\mathcal{K}(T)f$ preserves the order, i.e.,

\[ u_1 \leq_U u_2 \Rightarrow \mathcal{K}(T)f(u_1) \leq_{BV} \mathcal{K}(T)f(u_2) \]
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$$u_1 \leq_U u_2 \Rightarrow \mathcal{K}(T)f(u_1) \leq_{BTV} \mathcal{K}(T)f(u_2)$$
Some Possible Examples

- The first example is:
  
  \[ x \subseteq_{\text{Id}} y \iff x = y \]
  
  \[ x \subseteq_{\mathcal{P}} y \iff \forall e \in x \exists e' \in y . e \subseteq_{\text{Id}} e' \]

  The order \( \subseteq_{\mathcal{P}} \) captures the classical notion of nondeterministic reduction and can be turned into more specific cases. For example, the failure forcing variant \( \subseteq_{\mathcal{P}}^{E} \), where \( E \) stands for emptyset, guarantees that the first component fails no more than the second one. It is defined by replacing the clause for \( \subseteq_{\mathcal{P}} \) by

  \[ x \subseteq_{\mathcal{P}}^{E} y \iff (x = \emptyset \Rightarrow y = \emptyset) \land \forall e \in x \exists e' \in y . e \subseteq_{\text{Id}} e' \]

- Consider the partiality monad \( B = \text{Id} + 1 \). The set \( BU \) carries the familiar “flat” order:

  \[ x \subseteq_{B} y \iff x \neq \ast \Rightarrow x = y \land x = \ast \Rightarrow y = \ast \]
Forward and Backward Morphisms

- A possible (and intuitive) way of considering component $p$ as a refinement of another component $q$ is to consider that $p$-transitions are simply preserved in $q$. For example, for non-deterministic components this means set inclusion.

- Homomorphism can be used to relate two coalgebras.

\[
\begin{align*}
U & \xrightarrow{\alpha} \mathcal{K}(T)U \\
h & \downarrow \quad \downarrow \mathcal{K}(T)h \\
V & \xrightarrow{\beta} \mathcal{K}(T)V
\end{align*}
\]

- From homomorphisms we can only derive bisimulations!
- To build a witness for refinement relations, we separate the preservation and reflection aspects in homomorphism.
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Forward and Backward Morphisms

For a Kleisli category $\mathcal{K}(B)$ and two coalgebras $p = (U, \alpha : U \to \mathcal{K}(T)U)$ and $q = (V, \beta : V \to \mathcal{K}(T)V)$. A forward morphism $h : p \to q$ with respect to an order $\leq$ on $\mathcal{K}(T)$ is an arrow $h : U \to V$ such that

$$\mathcal{K}(T)h \cdot \alpha \leq \beta \cdot h$$

Dually, $h$ is called a backward morphism if the following conditions are satisfied:

$$\beta \cdot h \leq \mathcal{K}(T)h \cdot \alpha$$
Component Refinement

The existence of a forward (backward) morphism connecting two components \( p \) and \( q \) witnesses a refinement situation whose symmetric closure coincides, as expected, with bisimulation.

Component \( p \) is a behavior refinement of \( q \), written \( p \sqsubseteq_B q \), if there exist components \( r \) and \( s \) such that \( p \sim r \), \( q \sim s \) and a (seed preserving) forward morphism from \( r \) to \( s \).

A forward morphism is a “behavior preserving” mapping, but lying inside it is a more fundamental concept: to relate two coalgebras, one must show that all the transitions in one coalgebra are “mimicked” by the other. Such an intuition is formalized by the notion of \textit{simulation}. 
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Simulation

- For a given Kleisli category $K(B)$, a functor $T$ and a refinement preorder $\leq$, a lax relation lifting is an operation $\text{Rel}_{\leq}(K(T))$ mapping relation $R$ to $\leq \cdot \text{Rel}(K(T))(R) \cdot \leq$, where $\text{Rel}(K(T))(R)$ is the lifting of $R$ to $K(T)$ defined, as usual, as the $K(T)$-image of inclusion.

- Given coalgebras $(U, \alpha)$ and $(V, \beta)$, a simulation is a $\text{Rel}_{\leq}(K(T))$-coalgebra over $\alpha$ and $\beta$, i.e., a relation $R$ such that, for all $u \in U$, $v \in V$,

$$\quad (u, v) \in R \implies (\alpha u, \beta v) \in \text{Rel}_{\leq}(K(T))(R)$$

Diagram:

\[
\begin{array}{ccc}
U & \xleftarrow{\pi_1} & R & \xrightarrow{\pi_2} & V \\
\alpha \downarrow & & \downarrow & & \beta \\
\mathcal{K}(T)U & \xleftarrow{\mathcal{K}(T)\pi_1} & \text{Rel}(\mathcal{K}(T))(R) & \xrightarrow{\mathcal{K}(T)\pi_2} & \mathcal{K}(T)V
\end{array}
\]
Soundness and Completeness Results

For two coalgebras $p$ and $q$,

**Theorem (soundness)**

To prove $p \sqsubseteq_B q$ it is sufficient to exhibit a simulation $R$ relating $p$ and $q$.

**Theorem (completeness)**

If $p \sqsubseteq_B q$ and $h$ is the witness forward morphism, then $\sim \cdot \text{Graph}(h) \cdot \sim$ is a simulation between $p$ and $q$. 
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Pre-Galois Connection

For a Kleisli category $\mathcal{K}(B)$ and functor $T$, let $\leq$ be an order on $\mathcal{K}(T)$, a pre-Galois connection between two $\mathcal{K}(T)$-coalgebras $(U, \alpha)$ and $(V, \beta)$ is a pair of arrows $f : U \to V$ and $g : V \to U$, such that for all $u \in U$ and $v \in V,$

$$\alpha(u) \leq_{\mathcal{K}(T)U} \mathcal{K}(T)g \cdot \beta(v) \text{ iff } \mathcal{K}(T)f \cdot \alpha(u) \leq_{\mathcal{K}(T)V} \beta(v)$$

We say that $f$ is the lower adjoint and $g$ is the upper adjoint of the pre-Galois connection.
Composition and Identity for Pre-Galois Connections

If \((f, g)\) is a pre-Galois connection between two coalgebras \((U, \alpha : U \to \mathbb{H}(T)U)\) and \((V, \beta : V \to \mathbb{H}(T)V)\), and \((h, k)\) is a pre-Galois connection between two coalgebras \((V, \beta : V \to \mathbb{H}(T)V)\) and \((W, \gamma : W \to \mathbb{H}(T)W)\), then \((h \cdot f, g \cdot k)\) is a pre-Galois connection between \((U, \alpha : U \to \mathbb{H}(T)U)\) and \((W, \gamma : W \to \mathbb{H}(T)W)\).

\((\text{id}, \text{id})\) where \text{id} denotes the identity function on \(U\) is a pre-Galois connection between a coalgebra \((U, \alpha : U \to \mathbb{H}(T)U)\) and itself.
Composition and Identity for Pre-Galois Connections

- If \((f, g)\) is a pre-Galois connection between two coalgebras \((U, \alpha : U \to \mathcal{K}(T)U)\) and \((V, \beta : V \to \mathcal{K}(T)V)\), and \((h, k)\) is a pre-Galois connection between two coalgebras \((V, \beta : V \to \mathcal{K}(T)V)\) and \((W, \gamma : W \to \mathcal{K}(T)W)\), then \((h \cdot f, g \cdot k)\) is a pre-Galois connection between \((U, \alpha : U \to \mathcal{K}(T)U)\) and \((W, \gamma : W \to \mathcal{K}(T)W)\).

- \((\text{id}, \text{id})\) where \text{id} denotes the identity function on \(U\) is a pre-Galois connection between a coalgebra \((U, \alpha : U \to \mathcal{K}(T)U)\) and itself.
Cancellation

If we introduce an order $\preceq_U$ on $U$ for $(U, \alpha : U \to \mathcal{K}(T)U)$ as $u \preceq_U u'$ iff $\alpha(u) \preceq_{\mathcal{K}(T)U} \alpha(u')$, i.e., we assume that $\preceq$ reflects the transition structure $\to$. In other words, the functor $\mathcal{K}(T)$ is order-preserving, then

**Lemma (Cancellation)**

If $(f, g)$ is a pre-Galois connection between two coalgebras $(U, \alpha : U \to \mathcal{K}(T)U)$ and $(V, \beta : V \to \mathcal{K}(T)V)$, then we have

\[ f \cdot g \preceq_V \text{id}_V \text{ and } \text{id}_U \preceq_U g \cdot f \]

and

**Lemma**

If $(f, g)$ is a pre-Galois connection between two coalgebras $(U, \alpha : U \to \mathcal{K}(T)U)$ and $(V, \beta : V \to \mathcal{K}(T)V)$, then $f$ and $g$ are both monotonic with respect to $\preceq_U$ and $\preceq_V$. 
Cancellation

If we introduce an order \( \prec_U \) on \( U \) for \((U, \alpha : U \to \mathcal{H}(T)U)\) as \( u \prec_U u' \) iff \( \alpha(u) \leq \mathcal{H}(T) \alpha(u') \), i.e., we assume that \( \prec \) reflects the transition structure \( \to \). In other words, the functor \( \mathcal{H}(T) \) is order-preserving, then

**Lemma (Cancellation)**

*If \((f, g)\) is a pre-Galois connection between two coalgebras \((U, \alpha : U \to \mathcal{H}(T)U)\) and \((V, \beta : V \to \mathcal{H}(T)V)\), then we have*

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f \cdot g \prec_V \text{id}_V \text{ and } \text{id}_U \prec_U g \cdot f
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**Lemma**

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Cancellation

If we introduce an order $<_U$ on $U$ for $(U, \alpha : U \to \mathcal{K}(T)U)$ as $u<_U u'$ iff $\alpha(u) \leq \mathcal{K}(T)U \alpha(u')$, i.e., we assume that $<$ reflects the transition structure $\to$. In other words, the functor $\mathcal{K}(T)$ is order-preserving, then

**Lemma (Cancellation)**

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Relationship with Galois Connection

**Theorem**

If \((f, g)\) is a pre-Galois connection between two coalgebras \((U, \alpha : U \to \mathcal{H}(T)U)\) and \((V, \beta : V \to \mathcal{H}(T)V)\), for the orders \(<_U\) and \(<_V\) on \(U\) and \(V\), \((f, g)\) is a Galois connection.
Theorem

If \((f, g)\) is a pre-Galois connection between two coalgebras \((U, \alpha : U \to \mathcal{H}(T)U)\) and \((V, \beta : V \to \mathcal{H}(T)V)\), then \(f \cdot g \cdot f \sim f\) and \(g \cdot f \cdot g \sim g\).
Properties for Adjoints

The adjoints in a pre-Galois connection uniquely determine each other when the order $\leq$ is a partial order and $\mathcal{K}(T)$ is a faithful functor.

**Theorem**

If the order $\leq$ is a partial order, and $(f, g)$ and $(f, h)$ are pre-Galois connections between two coalgebras $(U, \alpha : U \to \mathcal{K}(T)U)$ and $(V, \beta : V \to \mathcal{K}(T)V)$ where $\mathcal{K}(T)$ is faithful, then $g = h$ (similarly for the dual case).

**Corollary**

If $\leq$ is a preorder, and $(f, g)$ and $(f, h)$ are pre-Galois connections between two coalgebras $(U, \alpha : U \to \mathcal{K}(T)U)$ and $(V, \beta : V \to \mathcal{K}(T)V)$ where $\mathcal{K}(T)$ is faithful, then $g \sim h$ (similarly for the dual case).
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- $f$ is monic iff $g$ is epic iff $g \cdot f = \text{id}_U$;
- $g$ is monic iff $f$ is epic iff $f \cdot g = \text{id}_V$.

**Corollary**

If $\leq$ is a preorder, and $(f, g)$ is a pre-Galois connection between $(U, \alpha : U \to \mathcal{K}(T)U)$ and $(V, \beta : V \to \mathcal{K}(T)V)$ where $\mathcal{K}(T)$ is faithful, then

- $f \, (g)$ is monic $\Rightarrow$ $g \cdot f \sim \text{id}_U$ ($f \cdot g \sim \text{id}_V$);
- $f \, (g)$ is epic $\Rightarrow$ $f \cdot g \sim \text{id}_V$ ($g \cdot f \sim \text{id}_U$).
Properties for Adjoint Properties

Theorem

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- \(f (g)\) is monic \(\Rightarrow g \cdot f \sim \text{id}_U\) \((f \cdot g \sim \text{id}_V)\);
- \(f (g)\) is epic \(\Rightarrow f \cdot g \sim \text{id}_V\) \((g \cdot f \sim \text{id}_U)\).
Given a pre-Galois connection \((f : U \rightarrow V, g : V \rightarrow U)\), we can extract the relation \(R_{(f,g)} \subseteq U \times V\) as follows:

\[
R_{(f,g)} = \{(u, v) \mid \mathcal{K}(T)f \cdot \alpha(u) \leq \mathcal{K}(T)V \beta(v)\}
\]

or equivalently

\[
R_{(f,g)} = \{(u, v) \mid \alpha(u) \leq \mathcal{K}(T)U \mathcal{K}(T)g \cdot \beta(v)\}
\]

**Theorem**

The relation \(R_{(f,g)}\) is a simulation.

**Corollary**

If the preorder \(\leq\) be equality =, then \(R_{(f,g)}\) is a bisimulation.
Linking Pre-Galois Connection with Refinement

Given a pre-Galois connection \((f : U \to V, g : V \to U)\), we can extract the relation \(R_{(f,g)} \subseteq U \times V\) as follows:

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R_{(f,g)} = \{(u, v) \mid \mathcal{K}(T)f \cdot \alpha(u) \leq \mathcal{K}(T)V \beta(v)\}
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or equivalently

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**Theorem**

The relation \(R_{(f,g)}\) is a simulation.

**Corollary**

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Conclusions

- The coalgebraic model for state based components is rebuilt in the Kleisli category.
- The refinement theory for generic state-based components is re-examined for coalgebras in the Kleisli category.
- The notion of pre-Galois connection is defined and some properties are proved.
Future work

- Go deeper into the concept itself
  - Existence of the adjoints in a pre-Galois connection
  - Completeness of pre-Galois connection for refinement
- Application of pre-Galois connection in refinement examples
Thank you!